Exercise 33

A thin wire is bent into the shape of a semicircle $x^2 + y^2 = 4$, $x \geq 0$. If the linear density is a constant k, find the mass and center of mass of the wire.

Solution

Begin by parameterizing the wire's position as a function of t: $x(t) = 2\cos t$ and $y(t) = 2\sin t$ with $-\pi/2 \leq t \leq \pi/2$. Integrate the density over the wire's length in order to get the wire's mass.

$$\begin{aligned}
m &= \int_C \rho\, ds \\
&= \int_{-\pi/2}^{\pi/2} k\sqrt{\left(\frac{dx}{dt}\right)^2 + \left(\frac{dy}{dt}\right)^2}\, dt \\
&= k\int_{-\pi/2}^{\pi/2} \sqrt{(-2\sin t)^2 + (2\cos t)^2}\, dt \\
&= k\int_{-\pi/2}^{\pi/2} \sqrt{4\sin^2 t + 4\cos^2 t}\, dt \\
&= k\int_{-\pi/2}^{\pi/2} \sqrt{4(\sin^2 t + \cos^2 t)}\, dt \\
&= k\int_{-\pi/2}^{\pi/2} \sqrt{4}\, dt \\
&= 2k\int_{-\pi/2}^{\pi/2} dt \\
&= 2\pi k.
\end{aligned}$$

Calculate the x-coordinate of the center of mass.

$$\begin{aligned}\bar{x} = \frac{\int x\,dm}{\int dm} = \frac{\int_C x(\rho\,ds)}{\int_C \rho\,ds} &= \frac{\rho\int_{-\pi/2}^{\pi/2} x(t)\sqrt{\left(\frac{dx}{dt}\right)^2+\left(\frac{dy}{dt}\right)^2}\,dt}{\rho\int_{-\pi/2}^{\pi/2}\sqrt{\left(\frac{dx}{dt}\right)^2+\left(\frac{dy}{dt}\right)^2}\,dt} \\ &= \frac{\int_{-\pi/2}^{\pi/2} 2\cos t\sqrt{(-2\sin t)^2+(2\cos t)^2}\,dt}{\int_{-\pi/2}^{\pi/2}\sqrt{(-2\sin t)^2+(2\cos t)^2}\,dt} \\ &= \frac{\int_{-\pi/2}^{\pi/2}(2\cos t)\sqrt{4}\,dt}{\int_{-\pi/2}^{\pi/2}\sqrt{4}\,dt} \\ &= \frac{4}{\pi}\end{aligned}$$

Calculate the y-coordinate of the center of mass.

$$\begin{aligned}\bar{y} = \frac{\int y\,dm}{\int dm} = \frac{\int_C y(\rho\,ds)}{\int_C \rho\,ds} &= \frac{\rho\int_{-\pi/2}^{\pi/2} y(t)\sqrt{\left(\frac{dx}{dt}\right)^2+\left(\frac{dy}{dt}\right)^2}\,dt}{\rho\int_{-\pi/2}^{\pi/2}\sqrt{\left(\frac{dx}{dt}\right)^2+\left(\frac{dy}{dt}\right)^2}\,dt} \\ &= \frac{\int_{-\pi/2}^{\pi/2} 2\sin t\sqrt{(-2\sin t)^2+(2\cos t)^2}\,dt}{\int_{-\pi/2}^{\pi/2}\sqrt{(-2\sin t)^2+(2\cos t)^2}\,dt} \\ &= \frac{\int_{-\pi/2}^{\pi/2}(2\sin t)\sqrt{4}\,dt}{\int_{-\pi/2}^{\pi/2}\sqrt{4}\,dt} \\ &= 0\end{aligned}$$

Therefore, the center of mass of the semicircular wire is

$$\left(\frac{4}{\pi}, 0\right).$$